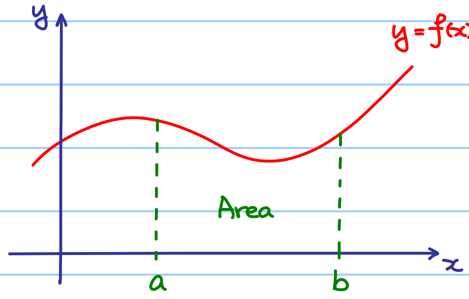
§7 Integration

7.1 Riemann Integral and Darboux Integral

Goal: Find the area of the region under the curve $y=f(x)$ over an interval $[a,b]$.



Idea: Approximated by rectangles. How?

We will introduce Riemann sums and Darboux sums, then two definitions of integrability will be given. However, it can be shown that the two definitions are equivalent.

Definition:

A partition of the interval $[a,b]$ is a finite set $P=\{x_0,x_1,\cdots,x_n\}$ such that

$$a=x_0<x_1<x_2<\cdots<x_n=b.$$

We denote $\Delta x_k = x_k - x_{k-1}$ for $k=1,2,\cdots,n$.

The norm (or mesh size) of P is defined by $\|P\| = \max\{\Delta x_k : k=1,2,\cdots,n\}$.

A tagged partition is a partition $P=\{x_0,x_1,\cdots,x_n\}$ of $[a,b]$ endowed with tags $\vec{c}=(c_1,c_2,\cdots,c_n)$ such that $c_k\in[x_{k-1},x_k]$ for $k=1,2,\cdots,n$, it is denoted by $(P,\vec{c})$.

Let P and P' be partitions of $[a,b]$.

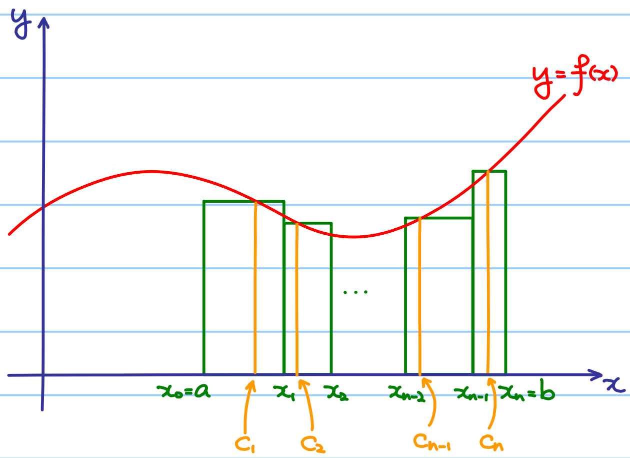
P' is said to be a refinement of P if $P\subset P'$.

$P\cup P'$ is said to be the common refinement of P and P'.

Definition :

Let $f:[a,b]\to\mathbb{R}$ and $(P,\vec{c})$ be a tagged partition of $[a,b]$.

The Riemann sum associated by f and $(P,\vec{c})$ is defined by $S(f,P,\vec{c})=\sum_{k=1}^{n} f(c_k)\Delta x_k$



Definition :

Let $f:[a,b]\to\mathbb{R}$ be a bounded function and let P be a partition of $[a,b]$.

Define $M_k=\sup f([x_{k-1},x_k])$

$m_k=\inf f([x_{k-1},x_k])$

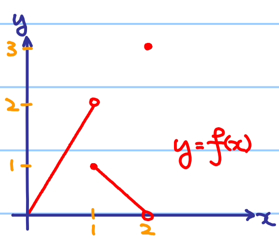
Then the Darboux upper sum is defined by $U(f,P)=\sum_{k=1}^{n} M_k\Delta x_k$ and

the Darboux lower sum is defined by $L(f,P)=\sum_{k=1}^{n} m_k\Delta x_k$.

Exercise :

1) Let $f:[0,2]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}2x & \text{if } 0\le x<1\\ 2-x & \text{if } 1\le x<2\\ 3 & \text{if } x=2\end{cases}$$



Let $P=\{0,1,\frac{3}{2},2\}$, $\vec{c}=(\frac{1}{3},1,\frac{7}{4})$

Find $S(f,P,\vec{c})$, $U(f,P)$ and $L(f,P)$.

Ans : $S(f,P,\vec{c})=(\frac{2}{3})(1)+(1)(\frac{1}{2})+(\frac{1}{4})(\frac{1}{2})$

$U(f,P)=(2)(1)+(1)(\frac{1}{2})+(3)(\frac{1}{2})$

$L(f,P)=(0)(1)+(\frac{1}{2})(\frac{1}{2})+(0)(\frac{1}{2})$

2) Let $f:[a,b]\to\mathbb{R}$ be a bounded function and let P be a partition of $[a,b]$.

Prove that $L(f,P)\le U(f,P)$.

Definition: (Riemann Integrable)

Let $f:[a,b]\to\mathbb{R}$.

f is said to Riemann Integrable on $[a,b]$ if

there exists $A\in\mathbb{R}$ such that for all $\varepsilon>0$, there exists $\delta>0$ such that

for all tagged partition $(P,\vec{z})$ with $\|P\|<\delta$, we have $|S(f,P,\vec{z})-A|<\varepsilon$.

$(\exists A\in\mathbb{R})(\forall\varepsilon>0)(\exists\delta>0)(\forall(P,\vec{z})$ with $\|P\|<\delta)(|S(f,P,\vec{z})-A|<\varepsilon)$

Meaning:

$A\in\mathbb{R}$ is supposed to be the "area".

No matter how small ε is, there exists a small δ such that if width of each rectangle is smaller than δ, then $S(f,P,\vec{z})$ must be a "good approximation" $(|S(f,P,\vec{z})-A|<\varepsilon)$.

Exercise:

Prove that if f is Riemann Integrable on $[a,b]$, the number A is uniquely detemined.

We denote A by $(R)\int_a^b f$ or $(R)\int_a^b f(x)dx$ and it is said to be the Riemann Integral.

Exercise:

Write down the negation of the above definition.

Ans: f is NOT Riemann integrable if

$(\forall A\in\mathbb{R})(\exists\varepsilon>0)(\forall\delta>0)(\exists(P,\vec{z})$ with $\|P\|<\delta)(|S(f,P,\vec{z})-A|\geq\varepsilon)$

Note, we do NOT assume the boundness of f in the definition, however we have

Theorem:

If f is Riemann Integrable on $[a,b]$, then it is bounded on $[a,b]$.

proof:

Suppose the f is NOT bounded above.

there exists a sequence $\{y_n\}\subseteq[a,b]$ such that $f(y_n)\geq n$ for all $n\in\mathbb{N}$.

Let $A\in\mathbb{R}$, take $\varepsilon=1$.

Let $\delta>0$ and take $N\in\mathbb{N}$ such that $N>\frac{b-a}{\delta}$, then let P be the even partition which divides $[a,b]$ into N equal subintervals, so $\|P\|=\frac{b-a}{N}<\delta$.

$(\Delta x=\frac{b-a}{N}$ and $x_k=a+k\Delta x$ for $k=1,2,\cdots,N.)$

Since $\{y_n\}$ is an infinite sequence, there exists $k \in \{1, 2, \cdots, N\}$ such that

$[x_{k-1}, x_k] \cap \{y_n\}$ is an infinite set.

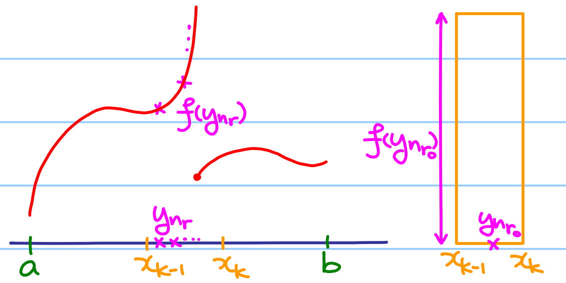
Let $\{y_{n_r}\} = [x_{k-1}, x_k] \cap \{y_n\}$ which is an infinite subsequence of $\{y_n\}$.

Take $c_j = x_j$ for $1 \le j \le N$, $j \ne k$.

$\exists\, n_{r_0} \in \mathbb{N}$ such that $n_{r_0} \geq -\sum_{j=1, j\neq k}^{n} f(x_j) + \frac{A}{\Delta x} - \frac{1}{\Delta x}$

Choose $c_k = y_{n_{r_0}}$

$$f(c_k)\Delta x = f(y_{n_{r_0}})\Delta x \geq n_{r_0}\Delta x \geq -\sum_{j=1, j\neq k}^{n} f(x_j)\Delta x + A\Delta x - 1$$



$$|S(f, P, \vec{c}) - A| = f(c_k)\Delta x + \sum_{j=1, j\neq k}^{n} f(x_j)\Delta x - A \geq 1$$

By choosing $y_{n_{r_0}}$,
$f(y_{n_{r_0}})(x_k - x_{k-1})$ will be larger than anything

Definition: (Darboux Integrable)

Let $f: [a,b] \to \mathbb{R}$ be a bounded function.

f is said to Darboux Integrable on $[a,b]$ if

there exists unique $A \in \mathbb{R}$ such that for all partition P, we have

$L(f,P) \le A \le U(f,P)$

Meaning:

For any partition P, it gives estimations $L(f,P)$ and $U(f,P)$, and

$L(f,P) \le A \le U(f,P)$ —(*)

underestimate overestimate

However, among all partitions, only one real number A satisfies (*)

If f is Darboux Integrable on $[a,b]$, then

we denote A by $(D)\int_a^b f$ or $(D)\int_a^b f(x)\,dx$ and it is said to be the Darboux Integral.

Lemma: (Refinement Lemma)

Suppose that $f:[a,b]\to\mathbb{R}$ is bounded.

Let P be a partition of $[a,b]$ and let P' be a refinement of P. Then

$L(f,P)\le L(f,P')\le U(f,P')\le U(f,P)$

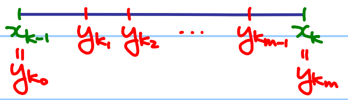
(Remark: Refinement $\Rightarrow$ Better approximation)

proof:

Claim: $U(f,P')\le U(f,P)$

Let $P=\{x_0,x_1,\cdots,x_n\}$ and

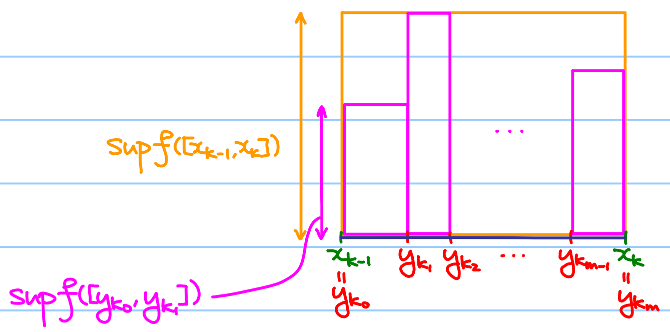
suppose that $y_{k_0},y_{k_1},\cdots,y_{k_m}\in P'\cap[x_{k-1},x_k]$ and $x_{k-1}=y_{k_0}<y_{k_1}<\cdots<y_{k_m}=x_k$



Then $\sup f([x_{k-1},x_k])\ge \sup f([y_{k_{i-1}},y_{k_i}])$ for $i=1,2,\cdots,m$

$\therefore \sum_{i=1}^{m}\sup f([y_{k_{i-1}},y_{k_i}])\cdot(y_{k_i}-y_{k_{i-1}})\le \sup f([x_{k-1},x_k])\cdot(x_k-x_{k-1})$

Summing up all inequalities from each $[x_{k-1},x_k]$, the result follows.



Exercise:

Let $f:[a,b]\to\mathbb{R}$ be a bounded function and let P_1,P_2 be partitions of $[a,b]$.

Prove that $L(f,P_1)\le U(f,P_2)$.

Hint: consider $P=P_1\cup P_2$ and the refinement lemma.

Therefore, let $\mathcal{U}=\{U(f,P): P \text{ is a partition}\}$ and $\mathcal{L}=\{L(f,P): P \text{ is a partition}\}$.

then $l\le u$ for all $l\in\mathcal{L}$ and $u\in\mathcal{U}$.

We denote $\inf\{U(f,P): P \text{ is a partition}\}$ and $\sup\{L(f,P): P \text{ is a partition}\}$

by $(D)\overline{\int_a^b} f$ and $(D)\underline{\int_a^b} f$ respectively.

Theorem:

Let $f:[a,b]\to\mathbb{R}$ be a bounded function. Then TFAE:

a) f is Darboux integrable on $[a,b]$;

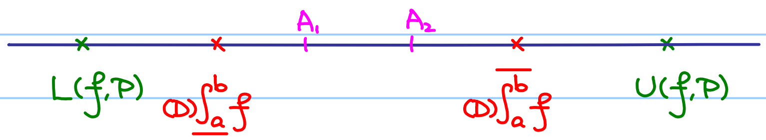
b) $(D)\overline{\int_a^b} f = (D)\underline{\int_a^b} f$;

c) For all $\varepsilon>0$, there exist a partition P of $[a,b]$ such that $U(f,P)-L(f,P)<\varepsilon$.

proof:

(a) $\Rightarrow$ (b): Suppose the contrary, $(D)\underline{\int_a^b} f < (D)\overline{\int_a^b} f$.

Then there exist A_1, A_2 such that $A_1 \neq A_2$



Therefore, $L(f,P) \le A_1, A_2 \le U(f,P)$ for all partition P

which contradicts to the assumption.

(b) $\Rightarrow$ (c): Let $\varepsilon>0$, there exist partitions P_1, P_2 such that

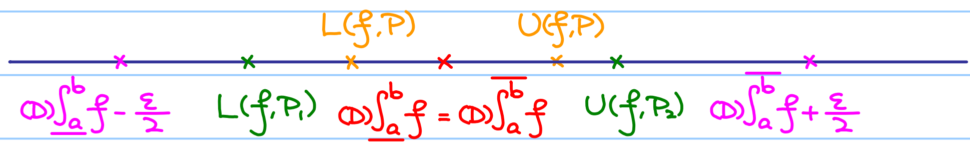
$$(D)\underline{\int_a^b} f - \frac{\varepsilon}{2} < L(f,P_1) \le (D)\underline{\int_a^b} f$$

$$\| \leftarrow \text{By assumption}$$

$$(D)\overline{\int_a^b} f \le U(f,P_2) < (D)\overline{\int_a^b} f + \frac{\varepsilon}{2}$$

Let $P = P_1 \cup P_2$, then by applying the refinement lemma (twice),

$$L(f,P_1) \le L(f,P) \le U(f,P) \le U(f,P_2)$$



Then $U(f,P) - L(f,P) < \varepsilon$.

(c) $\Rightarrow$ (a): Suppose the contrary, there exists A_1, A_2 such that

$A_1 < A_2$ and $L(f,P) \le A_1 < A_2 \le U(f,P)$ for all partition P.

Take $\varepsilon = \frac{1}{2}(A_2 - A_1) > 0$, then

for all partition P, $U(f,P) - L(f,P) \ge A_2 - A_1 > \varepsilon$

which contradicts to the assumption.

7.2 Equivalence of Riemann Integrability and Darboux Integrability

Lemma:

Suppose that $f:[a,b]\to\mathbb{R}$ is bounded. Let P_0 be a partition of $[a,b]$.

Then for all $\varepsilon>0$, there exists $\delta>0$ such that for all partition P of $[a,b]$ with $\|P\|<\delta$,

we have $0\le U(f,P)-U(f,P\cup P_0)<\varepsilon$ and $0\le L(f,P\cup P_0)-L(f,P)<\varepsilon$.

(Remark: Fix a partition P_0 and let P be an arbitrary partion, by the refinement lemma,

$$\underbrace{L(f,P)\le L(f,P_0\cup P)}_{\text{difference?}}\le \underbrace{U(f,P_0\cup P)\le U(f,P)}_{\text{difference?}}$$

The difference (ε) can be arbitrarily small by choosing δ and restricting $\|P\|<\delta$.)

proof:

Let $P_0=\{x_0,x_1,x_2,\cdots,x_n\}$. If $n=1$, then $P_0=\{a,b\}$ and $P\cup P_0=P$, so the statement is trivial.

Therefore, it suffices to consider $n>1$.

Let $\varepsilon>0$, $C=\sup\{|f(x)|: a\le x\le b\}+1$,

Take $\delta=\min\left\{\dfrac{\varepsilon}{2(n-1)C},\Delta x_1,\Delta x_2,\cdots,\Delta x_n\right\}=\min\left\{\dfrac{\varepsilon}{2(n-1)C},\min\limits_{1\le k\le n}\{\Delta x_k\}\right\}>0$

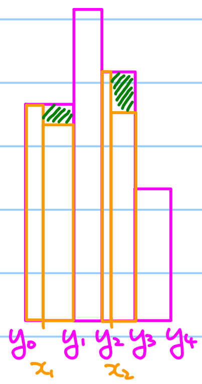
Let $P=\{y_0,y_1,\cdots,y_m\}$ be a partition of $[a,b]$ such that $\|P\|<\delta$, i.e. $y_i-y_{i-1}<\delta$, $i=1,2,\cdots,m$.

Since $\|P\|<\delta<\|P_0\|$, so for $k=1,2,\cdots,n-1$, there exist distinct m_k such that $x_k\in[y_{m_k-1},y_{m_k})$.

(Otherwise, $x_{k-1},x_k\in[y_{i-1},y_i)$

and $x_k-x_{k-1}=\Delta x_k<y_i-y_{i-1}\le\|P\|<\delta$

which contradicts to that $\delta\le\Delta x_k$)



▨ $=U(f,P)-U(f,P\cup P_0)$

$U(f,P)-U(f,P\cup P_0)$

$=\sum\limits_{k=1}^{n-1}\left[\sup f([y_{m_k-1},y_{m_k}])\cdot(y_{m_k}-y_{m_k-1})-\sup f([y_{m_k-1},x_k])\,(x_k-y_{m_k-1})-\sup f([y_{m_k}-x_k])\cdot(y_{m_k}-x_k)\right]$

$\le\sum\limits_{k=1}^{n-1}\left[|\sup f([y_{m_k-1},y_{m_k}])|\,(y_{m_k}-y_{m_k-1})+|\sup f([y_{m_k-1},x_k])|\,(x_k-y_{m_k-1})+|\sup f([y_{m_k}-x_k])|\,(y_{m_k}-x_k)\right]$

$\le 2(n-1)C\|P\|$

$<2(n-1)C\delta$

$<\varepsilon$

Theorem: (Riemann Integrability = Darboux Integrability)

Let $f:[a,b]\to\mathbb{R}$. Then

f is Riemann integrable on $[a,b]$ if and only if f is Darboux integrable on $[a,b]$.

Moreover, $(R)\int_a^b f = (D)\int_a^b f$.

proof:

$(R)\Rightarrow(D)$: Suppose that f is Riemann integrable. Then f is bounded.

Let $A=(R)\int_a^b f$.

Let $\varepsilon>0$, $\exists\,\delta>0$ such that for all P with $\|P\|<\delta$, we have $|S(f,P,\vec{c})-A|<\frac{\varepsilon}{4}$

Fix a partition P_0 with $\|P_0\|<\min\{\delta,1\}$. Let $P_0=\{x_0,x_1,\cdots,x_n\}$

Claim: There exist $\vec{c}_1,\vec{c}_2$ such that

$U(f,P_0)\geq S(f,P_0,\vec{c}_1)>U(f,P_0)-\frac{\varepsilon}{4}$ and

$L(f,P_0)\leq S(f,P_0,\vec{c}_2)<L(f,P_0)+\frac{\varepsilon}{4}$

proof of the claim:

For $k=1,2,\cdots,n$, there exists $c_k\in[x_{k-1},x_k]$ such that

$\sup f([x_{k-1},x_k])-\frac{\varepsilon}{4n}<f(c_k)\leq\sup f([x_{k-1},x_k])$

$\sup f([x_{k-1},x_k])\,\Delta x_k-\frac{\varepsilon}{4n}\overset{(*)}{<}f(c_k)\,\Delta x_k\leq\sup f([x_{k-1},x_k])\,\Delta x_k$ Note: $\Delta x_k\leq\|P_0\|\leq 1$

$\sum_{k=1}^{n}[\sup f([x_{k-1},x_k])\,\Delta x_k-\frac{\varepsilon}{4n}]<\sum_{k=1}^{n}f(c_k)\,\Delta x_k\leq\sum_{k=1}^{n}\sup f([x_{k-1},x_k])\,\Delta x_k$

$U(f,P_0)-\frac{\varepsilon}{4}<S(f,P_0,\vec{c}_1)\leq U(f,P_0)$ where $\vec{c}_1=(c_1,c_2,\cdots,c_n)$

Similar for the second inequality.

$$\begin{aligned}\text{Now, } U(f,P_0)-L(f,P_0)&<(S(f,P_0,\vec{c}_1)+\tfrac{\varepsilon}{4})-(S(f,P_0,\vec{c}_2)-\tfrac{\varepsilon}{4})\\&\leq|S(f,P_0,\vec{c}_1)-S(f,P_0,\vec{c}_2)|+\tfrac{\varepsilon}{2}\\&\leq|S(f,P_0,\vec{c}_1)-A|+|S(f,P_0,\vec{c}_2)-A|+\tfrac{\varepsilon}{2}\\&<\tfrac{\varepsilon}{4}+\tfrac{\varepsilon}{4}+\tfrac{\varepsilon}{2}=\varepsilon\end{aligned}$$

(D) $\Rightarrow$ (R) : Suppose that f is Darboux integrable. Let $A = (D)\int_a^b f$.

Let $\varepsilon > 0$, there exist a partition P_0 such that $U(f,P_0) - L(f,P_0) < \frac{\varepsilon}{2}$

By the previous lemma, there exists $\delta > 0$ such that

for all partition P of $[a,b]$ with $\|P\| < \delta$,

we have $0 \le U(f,P) - U(f,P\cup P_0) < \frac{\varepsilon}{2}$ and $0 \le L(f,P\cup P_0) - L(f,P) < \frac{\varepsilon}{2}$.

Consider a tagged partition $(P,\vec{c})$ with $\|P\| < \delta$.

Then,
$$\begin{cases} S(f,P,\vec{c}) \le U(f,P) < U(f,P\cup P_0) + \frac{\varepsilon}{2} \le U(f,P_0) + \frac{\varepsilon}{2} \\ S(f,P,\vec{c}) \ge L(f,P) > L(f,P\cup P_0) - \frac{\varepsilon}{2} > L(f,P_0) - \frac{\varepsilon}{2} \end{cases}$$

$$\begin{cases} S(f,P,\vec{c}) - A \le U(f,P_0) - A + \frac{\varepsilon}{2} < U(f,P_0) - L(f,P_0) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \\ S(f,P,\vec{c}) - A \ge L(f,P_0) - A - \frac{\varepsilon}{2} > L(f,P_0) - U(f,P_0) - \frac{\varepsilon}{2} < -\frac{\varepsilon}{2} - \frac{\varepsilon}{2} = -\varepsilon \end{cases}$$

$\therefore\ |S(f,P,\vec{c}) - A| < \varepsilon$

Exercise :

1) Let $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } 0 < x \le 1 \end{cases}$

Show that f is Riemann / Darboux integrable and the integral is 1.

2) Let $f : [0,1] \to \mathbb{R}$ defined by $f(x) = x$.

Show that f is Riemann / Darboux integrable and the integral is $\frac{1}{2}$.

3) Let $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

Show that f is not Riemann / Darboux integrable.

By showing the equivalence of Riemann integrability and Darboux integrability, we will simply say f is integrable on $[a,b]$. Furthermore, if we want to prove a statement involving integrability, we can freely choose either approach.

7.3 Criteria for Integrability

Theorem :

Suppose that $f:[a,b]\to\mathbb{R}$ is bounded. Then TFAE :

a) f is integrable on $[a,b]$

b) for all $\varepsilon>0$, there exists $\delta>0$ such that for all partition P of $[a,b]$ with $\|P\|<\delta$,

$U(f,P)-L(f,P)<\varepsilon$

c) if $\{P_n\}$ is any sequence of $[a,b]$ such that $\lim_{n\to\infty}\|P_n\|=0$, then

$\lim_{n\to\infty}L(f,P_n)=\lim_{n\to\infty}U(f,P_n)$.

proof :

(a)$\Rightarrow$(b) : Let $\varepsilon>0$, there exist a partition P_0 such that $U(f,P_0)-L(f,P_0)<\frac{\varepsilon}{2}$

By the previous lemma, there exists $\delta>0$ such that

for all partition P of $[a,b]$ with $\|P\|<\delta$,

we have $0\le U(f,P)-U(f,P\cup P_0)<\frac{\varepsilon}{4}$ and $0\le L(f,P\cup P_0)-L(f,P)<\frac{\varepsilon}{4}$.

Then $0\le U(f,P)<U(f,P\cup P_0)+\frac{\varepsilon}{4}<U(f,P_0)+\frac{\varepsilon}{4}$

$0\le -L(f,P)<-L(f,P\cup P_0)+\frac{\varepsilon}{4}<-L(f,P_0)+\frac{\varepsilon}{4}$

$\therefore U(f,P)-L(f,P)<(U(f,P_0)+\frac{\varepsilon}{4})+(-L(f,P_0)+\frac{\varepsilon}{4})<\varepsilon$

Exercise :

Prove (b)$\Rightarrow$(c) and (c)$\Rightarrow$(a).

Theorem :

Suppose that $f:[a,b]\to\mathbb{R}$ is monotone (i.e. either increasing or decreasing).

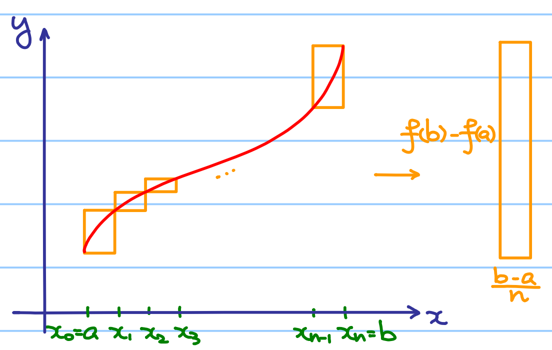
Then f is integrable on $[a,b]$.

proof :

Consider P_n to be the partition that divides $[a,b]$ into n equal subintervals.

Check : $U(f,P)-L(f,P)=\left(\frac{b-a}{n}\right)(f(b)-f(a))$

then $\lim_{n\to\infty}\|P_n\|=0$ and $\lim_{n\to\infty}L(f,P_n)=\lim_{n\to\infty}U(f,P_n)$.



Theorem :

Suppose that $f:[a,b]\to\mathbb{R}$ is continuous. Then f is integrable on $[a,b]$.

proof :

Note : f is uniformly continuous on $[a,b]$.

For all $\varepsilon>0$, there exists $\delta>0$ such that for all $u,v\in[a,b]$ with $|u-v|<\delta$,

we have $|f(u)-f(v)|<\frac{\varepsilon}{b-a}$

Choose $P=\{x_0,x_1,\cdots,x_n\}$ with $\|P\|<\delta$.

By the continuity of f and the max-min theorem,

for $k=1,2,\cdots,n$, there exists $x_{M_k}, x_{m_k}\in[x_{k-1},x_k]$ such that $f(x_{m_k})\le f(x)\le f(x_M)$ for all $x\in[x_{k-1},x_k]$

Then $U(f,P)-L(f,P)$

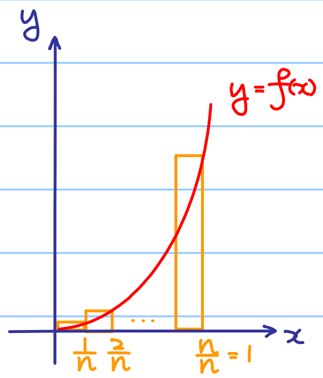
$$
\begin{aligned}
&=\sum_{k=1}^{n}[\sup f([x_{k-1},x_k])-\inf f([x_{k-1},x_k])]\,\Delta x_k\\
&=\sum_{k=1}^{n}[f(x_{M_k})-f(x_{m_k})]\,\Delta x_k\\
&\le\sum_{k=1}^{n}\frac{\varepsilon}{b-a}\cdot\Delta x_k \qquad (|x_{M_k}-x_{m_k}|\le\Delta x_k<\delta \Rightarrow |f(x_{M_k})-f(x_{m_k})|<\tfrac{\varepsilon}{b-a})\\
&=\varepsilon \qquad (\because \textstyle\sum_{k=1}^{n}\Delta x_k=b-a)
\end{aligned}
$$

Recall :

If $f:[0,1]\to\mathbb{R}$ is continuous.

We compute the limit $\lim_{n\to\infty}\sum_{i=1}^{n}f(\frac{i}{n})\frac{1}{n}$ by

$\lim_{n\to\infty}\sum_{i=1}^{n}f(\frac{i}{n})\frac{1}{n}=\int_0^1 f$, why?



Since f is continuous on $[0,1]$,

f is integrable on $[0,1]$.

Let $P_n=\{0,\frac{1}{n},\frac{2}{n},\cdots,1\}$, then

$$L(f,P_n)\le\sum_{i=1}^{n}f(\tfrac{i}{n})\tfrac{1}{n}\le U(f,P_n)$$

but $\lim_{n\to\infty}L(f,P_n)=\lim_{n\to\infty}U(f,P_n)=\int_0^1 f$,

so by the sandwich theorem, $\lim_{n\to\infty}\sum_{i=1}^{n}f(\frac{i}{n})\frac{1}{n}=\int_0^1 f$

e.g. $\lim_{n\to\infty}\frac{1}{n}(e^{1/n}+e^{2/n}+e^{3/n}+\cdots+e^{n/n})=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}e^{i/n}$

$=\int_0^1 e^x\,dx$

$(=[e^x]_0^1=e-1)$ Why?

7.4 Linearity, Monotonicity and Additivity

Theorem : (Linearity)

Suppose that $f, g : [a,b] \to \mathbb{R}$ are integrable and let $\alpha \in \mathbb{R}$. Then

1) αf is integrable and $\int_a^b \alpha f = \alpha \int_a^b f$;

2) $f+g$ is integrable and $\int_a^b f+g = \int_a^b f + \int_a^b g$.

proof :

Exercise :

If $I = [c,d]$, show that

a) $\sup \alpha f(I) - \inf \alpha f(I) \leq |\alpha| (\sup f(I) - \inf f(I))$

b) $\sup (f+g)(I) = \sup f(I) + \sup g(I)$ and $\inf (f+g)(I) = \inf f(I) + \inf g(I)$

1) Let $\varepsilon > 0$, there exists partition $P = \{x_0, x_1, x_2, \cdots, x_n\}$ of $[a,b]$ such that $U(f,P) - L(f,P) < \frac{\varepsilon}{|\alpha|}$

Then,
$$
\begin{aligned}
U(\alpha f, P) - L(\alpha f, P) &= \sum_{i=1}^{n} [\sup \alpha f([x_{i-1}, x_i]) - \inf \alpha f([x_{i-1}, x_i])] (x_i - x_{i-1}) \\
&\leq \sum_{i=1}^{n} |\alpha| (\sup f([x_{i-1}, x_i]) - \inf f([x_{i-1}, x_i])] (x_i - x_{i-1}) \\
&= |\alpha| (U(f,P) - L(f,P)) \\
&< \varepsilon
\end{aligned}
$$

Exercise :

Prove (2).

Remarks :

1) Suppose that $f, g : [a,b] \to \mathbb{R}$ are integrable and let $\alpha, \beta \in \mathbb{R}$.
Then $\alpha f + \beta g$ is integrable on $[a,b]$ and $\int_a^b \alpha f + \beta g = \alpha \int_a^b f + \beta \int_a^b g$.

2) Reconstruct the proof of the above theorem in Riemann's approach.

Theorem : (Monotonicity)

Suppose that $f : [a,b] \to \mathbb{R}$ is integrable and $f(x) \geq 0$ for all $x \in [a,b]$.

proof :

For every partition P of $[a,b]$, we have $L(f,P) \geq 0$.

Recall : $\int_a^b f = \sup \{ L(f,P) : P \text{ is a partition of } [a,b] \}$

Fix a partition P_0, then $\int_a^b f \geq L(f, P_0) \geq 0$

Remark:

1) Suppose that $f, g : [a,b] \to \mathbb{R}$ are integrable and $f \geq g$ on $[a,b]$. Then $\int_a^b f \geq \int_a^b g$.

2) Reconstruct the proof of the above theorem in Riemann's approach.

Theorem:

Let $a<c<b$. Then the function $f:[a,b]\to\mathbb{R}$ is integrable if and only if both $f:[a,c]\to\mathbb{R}$ and $f:[c,b]\to\mathbb{R}$ are integrable, in which case $\int_a^b f = \int_a^c f + \int_c^b f$.

proof:

"$\Rightarrow$" Suppose that $f:[a,b]\to\mathbb{R}$ is integrable.

Let $\varepsilon>0$,

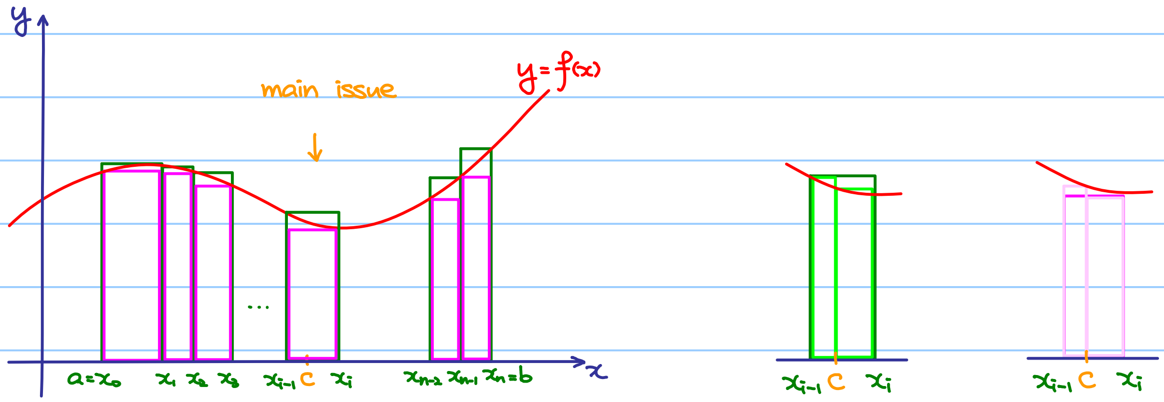
there exists partition $P'=\{x_0, x_1, x_2, \cdots, x_n\}$ of $[a,b]$ such that $U(f,P')-L(f,P')<\varepsilon$

let $P = P' \cup \{c\}$, $P_1 = [a,c]\cap P$ and $P_2 = [c,b]\cap P$.

P_1 and P_2 are partitions of $[a,c]$ and $[c,b]$ respectively.

By the refinement lemma, $U(f,P)-L(f,P) \leq U(f,P')-L(f,P')<\varepsilon$

Then, $U(f,P_1)-L(f,P_1) \leq U(f,P)-L(f,P)<\varepsilon$ and $U(f,P_2)-L(f,P_2)\leq U(f,P)-L(f,P)<\varepsilon$



$\inf f([x_{i-1},x_i]) \leq \inf f([x_{i-1},x_i]) + \inf f([x_{i-1},x_i]) \leq \sup f([x_{i-1},x_i]) + \sup f([x_{i-1},x_i]) \leq \sup f([x_{i-1},x_i])$

Exercise:

Prove the converse.

(Hint: If P_1 and P_2 are partitions of $[a,c]$ and $[c,b]$ respectively,

then $P = P_1 \cup P_2$ is a partition of $[a,b]$.)

Definition :

1) Suppose that $f:[a,b]\to\mathbb{R}$. We define $\int_a^a f = 0$.

2) Suppose that $f:[a,b]\to\mathbb{R}$ is integrable. We define $\int_b^a f = -\int_a^b f$.

Once we have the above definition, we can extend the additivity as follows :

Theorem :

Suppose that $f:[a,b]\to\mathbb{R}$ is integrable. Then for any $x_1, x_2, x_3 \in [a,b]$,
$\int_{x_1}^{x_3} f = \int_{x_1}^{x_2} f + \int_{x_2}^{x_3} f$.

7. Fundamental Theorem of Calculus

Theorem : (The First Fundamental Theorem of Calculus)

Let $f:[a,b]\to\mathbb{R}$ be integrable. Suppose that the function $F:[a,b]\to\mathbb{R}$ is continuous, that $F:(a,b)\to\mathbb{R}$ is differentiable and that $F'(x)=f(x)$ for all $x\in(a,b)$.

Then $\int_a^b f = F(b)-F(a)$.

proof :

It suffices to show $L(f,P) \le F(b)-F(a) \le U(f,P)$ for all partition P of $[a,b]$.

Let $P=\{x_0,x_1,\cdots,x_n\}$ be a partition of $[a,b]$.

Apply the mean value theorem to F on each $[x_{i-1},x_i]$ for $i=1,2,\cdots,n$,

there exist $c_i\in(x_{i-1},x_i)$ such that
$$\begin{aligned} F(x_i)-F(x_{i-1}) &= F'(c_i)(x_i-x_{i-1}) \\ &= f(c_i)(x_i-x_{i-1}) \end{aligned}$$

Note that $\inf f([x_{i-1},x_i]) \le f(c_i) \le \sup f([x_{i-1},x_i])$

$$\inf f([x_{i-1},x_i])(x_i-x_{i-1}) \le f(c_i)(x_i-x_{i-1}) \le \sup f([x_{i-1},x_i])(x_i-x_{i-1})$$

$$\sum_{i=1}^{n} \inf f([x_{i-1},x_i])(x_i-x_{i-1}) \le \sum_{i=1}^{n} F(x_i)-F(x_{i-1}) \le \sum_{i=1}^{n} \sup f([x_{i-1},x_i])(x_i-x_{i-1})$$

$$L(f,P) \le F(b)-F(a) \le U(f,P)$$

Example :

Let $f:[0,1]\to\mathbb{R}$ defined by $f(x)=x^2$

f is continuous on $[0,1]$ $\Rightarrow$ f is integrable on $[0,1]$

Furthermore, let $F:[0,1]\to\mathbb{R}$ defined by $F(x)=\frac{1}{3}x^3$

Then f is continuous on $[0,1]$ and $F'(x)=f(x)$ on $(0,1)$.

$\therefore$ By the first fundamental theorem of calculus, $\int_0^1 f = F(1)-F(0)=\frac{1}{3}$.

However, consider the following case :

Let $f:[0,2]\to\mathbb{R}$ defined by $f(x)=\begin{cases}1 & \text{if } 0\le x<1\\ 2 & \text{if } 1\le x\le 2\end{cases}$

Show that f is integrable and the integral is 3.

However, does it exist $F:[0,1]\to\mathbb{R}$ such that

F is continuous on $[0,2]$ and $F'(x)=f(x)$ on $(0,2)$?

Unfortunately, the answer is negative. (Why?)

However, we still have the following result :

Theorem :

Let $f:[a,b]\to\mathbb{R}$ be integrable. Define $F:[a,b]\to\mathbb{R}$ by $F(x)=\int_a^x f$ (Why F is well-defined?)

Then $F(x)$ is continuous on $[a,b]$.

proof :

f is integrable on $[a,b]$ $\Rightarrow$ f is bounded on $[a,b]$

i.e. there exists $M>0$ such that $-M\le f(x)\le M$ for all $x\in[a,b]$.

Claim : $|F(u)-F(v)|\le M|u-v|$ for all $u,v\in[a,b]$, i.e. F is a Lipschitz Function on $[a,b]$.

Then, F is continuous (uniformly continuous in fact) on $[a,b]$.

proof of claim : WLOG, let $u>v$, $F(u)-F(v)=\int_u^v f$

$$-M\le f(x)\le M \quad \text{for all } x\in[v,u]$$

$$\int_u^v -M \le \int_u^v f(x) \le \int_u^v M$$

$$-M(u-v)\le F(u)-F(v)\le M(u-v)$$

$$\therefore\ |F(u)-F(v)|\le M|u-v|$$

Example :

Let $f:[0,2]\to\mathbb{R}$ defined by $f(x)=\begin{cases}1 & \text{if } 0\le x<1\\ 2 & \text{if } 1\le x\le 2\end{cases}$

$F(x)=\int_0^x f=\begin{cases}x & \text{if } 0\le x\le 1\\ 2x-1 & \text{if } 1<x\le 2\end{cases}$ which is continuous on $[0,2]$.

Question :

1) What is the relation between $F(x)=\int_a^x f$ and $f(x)$?

2) The first fundamental theorem of calculus only requires that f is integral on $[a,b]$.
How about we put a stronger assumption that f is continuous on $[a,b]$?

Both question can be answered by :

Theorem : (The Second Fundamental Theorem of Calculus)

Let $f:[a,b]\to\mathbb{R}$ be continuous. Define $F:[a,b]\to\mathbb{R}$ by $F(x)=\int_a^x f$

$F:(a,b)\to\mathbb{R}$ is differentiable and that $F'(x)=f(x)$ for all $x\in(a,b)$.

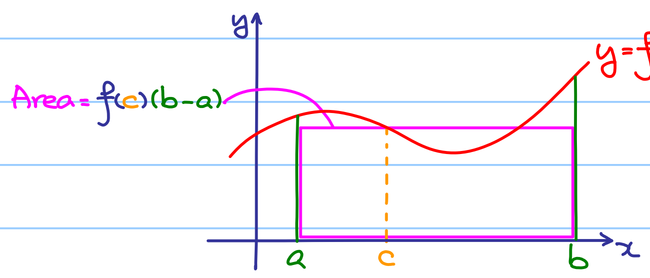
(If f is continuous on $[a,b]$,
then the area function $F(x)=\int_a^x f$, $x\in[a,b]$, is an antiderivative of f on (a,b).)

For the proof of the first fundamental theorem of calculus, we need

Theorem : (The Mean Value Theorem for Integrals)

Suppose that $f:[a,b]\to\mathbb{R}$ is continuous.

Then there exists $c\in[a,b]$ such that $\int_a^b f=f(c)(b-a)$.



proof :

f is continuous on $[a,b]$

$\Rightarrow$ there exist $x_m, x_M\in[a,b]$ such that $f(x_m)\le f(x)\le f(x_M)$ for all $x\in[a,b]$.

$$f(x_m) \leq f(x) \leq f(x_M) \qquad \text{all } x \in [v,u]$$

$$\int_a^b f(x_m) \leq \int_a^b f \leq \int_a^b f(x_M)$$

$$f(x_m) \leq \frac{1}{b-a}\int_a^b f \leq f(x_M)$$

By the intermediate value theorem, there exists c between x_m and x_M such that $f(c) = \frac{1}{b-a}\int_a^b f$, i.e. $\int_a^b f = f(c)(b-a)$.

proof of the first fundamental theorem of calculus :

$$\lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} f - \int_a^x f}{\Delta x}$$

By the mean value theorem for integrals

$$\int_a^{x+\Delta x} f - \int_a^x f = \int_x^{x+\Delta x} f$$

$$= \lim_{\Delta x \to 0} \frac{f(c)\,\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} f(c)$$

if $\Delta x > 0$, $\int_x^{x+\Delta x} f = f(c)\,\Delta x$ for some $c \in [x, x+\Delta x]$

$$= \lim_{c \to x} f(c)$$

if $\Delta x < 0$, $\int_x^{x+\Delta x} f = -\int_{x+\Delta x}^{x} f = -(f(c)(-\Delta x)) = f(c)\,\Delta x$

for some $c \in [x, x+\Delta x]$

$$= f(x)$$